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RIEMANN WAVES IN AN ELASTIC MEDIUM WITH SMALL ANISOTROPY†

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Riemann waves in an arbitrary elastic medium with small anisotropy of arbitrary form on the wave front plane are considered. The variation of the wave quantities is not assumed to be small. The principal attention is devoted to situations in which the presence of small anisotropy leads to qualitative modifications of the relations between the wave quantities and determines the tendency of the waves to break.

1. WE CONSIDER plane one-dimensional waves propagating in an elastic medium. We shall assume that the deformation and velocity of the medium are functions of the time t and the x coordinate of an orthogonal Cartesian system of Lagrange coordinates x_1, x_2, $x_3 = x$ connected with a certain initial state.

To describe the deformation in a one-dimensional process, we shall use the components of the tensor $\partial w_i / \partial x_k$, $i, k = 1, 2, 3$, of displacement gradients (w_i denotes the displacement vector), all the components of this tensor, except for $\partial w_i / \partial x \equiv u_i(x, t)$ being constant for a plane wave, $\partial w_i / \partial x_\alpha = \text{const}$, $\alpha = 1, 2$. In terms of the Lagrange variables, the system of equations for the process under consideration has the form [1]

$$\rho_0 \frac{\partial \upsilon_i}{\partial t} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial u_i}, \quad \frac{\partial \upsilon_i}{\partial x} = \frac{\partial u_i}{\partial t}, \quad \frac{\partial S}{\partial t} = 0, \quad i = 1,2,3 \tag{1.1}$$

Here ρ_0 is the density in the initial state, $\Phi(u_i, S)$ is the elasticity potential of the medium (the internal energy per unit Lagrange volume), and S is the entropy per unit mass. Henceforth we will assume that the medium is uniform and Φ does not depend directly on the Lagrange variable x.

For system (1.1) we shall study Riemann waves (simple waves), i.e. solutions of the form $u_i = u_i(\Phi(x, t))$, $\upsilon_i = \upsilon_i(\varphi(x, t))$, $S = \text{const}$, where $\varphi(x, t)$ is a function unknown in advance. This converts (1.1) into the system of differential equations

$$\begin{gathered}(\Phi_{ij} - \rho_0 c^2 \delta_{ij}) \frac{du_i}{d\varphi} = 0 \\ \frac{\partial \varphi}{\partial t} + c\frac{\partial \varphi}{\partial x} = 0, \quad \Phi_{ij} = \frac{\partial^2 \Phi}{\partial u_i \partial u_j}\end{gathered} \tag{1.2}$$

System (1.2) has non-trivial solutions if $\det \| \Phi_{ij} - \rho_0 c^2 \delta_{ij} \| = 0$, i.e. if $\rho_0 c^2 = \lambda$ are the eigenvalues of the symmetric matrix $\| \Phi_{ij} \|$, which are assumed to be positive, the latter being

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necessary in order for the medium to be stable. The three eigenvalues λ_k define the characteristic velocities c_k, while the eigenvectors $du_i/d\varphi$ define the tangent directions of three families of integral curves of system (1.2) for simple waves at every point of the u_i space, the families of curves being orthogonal to each other.

Since $c_k(u_i)$ represents the propagation velocity of the "phase" of a wave, the substitution $\lambda = \rho_0 c^2$ along an integral curve defines the variation of the shape of the wave profile with time. The variation of the wave profile with time changes direction at the extremum points of λ on the integral curve (increasing steepness changes into decreasing steepness, or the other way round). As a result of increasing steepness, the wave may "break" and develop a discontinuity.

The purpose of the present paper is to study the integral curves of system (1.2) and the variation of the characteristic velocities $c_k(u_i)$ along them, including the search for the extremum points of λ_k. The integral curves of the Riemann waves and the extremum points of λ_k depend on the form of $\Phi(u_k)$, i.e. on the properties of the medium. The dependence on the entropy will henceforth be ignored, since $s = \text{const}$ for every Riemann wave. If the elastic medium is isotropic, the components u_1 and u_2 characterizing the shear deformations in the plane of the wave front occur in the elastic potential only through the combination $r^2 = u_1^2 + u_2^2$.

The latter property may also be valid for some anisotropic media, the behaviour of which under displacement in the plane of the wave is isotropic, and which can be characterized by a vector $\mathbf{r}\{u_1, u_2\}$ lying in the plane of the wave front. We refer to this property as wave anisotropy. The purpose of this paper is to study the waves when there are small deviations from wave isotropy.

We will first mention some general properties of the integral curves of Riemann waves and the behaviour of the characteristic velocities along the curves in the case of wave isotropy $\Phi = F(r^2, u_3)$. Since $F(r^2, u_3)$ is a symmetric function with respect to any plane passing through the u_3 axis in the $u_1u_2u_3$ space, all derivatives of odd order in the direction normal to the plane turn out to be equal to zero in the plane. Clearly, one only needs to consider the behaviour of the eigenvalues and eigenvectors, for example, at the points of the u_1u_3 plane. Since, in view of the above, $F_{12} = F_{32} = 0$ (here and henceforth we set $F_{ij} = \partial^2 F/\partial u_i \partial u_j$), it follows that two of the eigenvectors of the matrix F_{ij} lie in the u_1u_3 plane, while the third one is normal to that plane. This means that two of the families of integral curves consist of curves lying in the planes passing through the u_3 axis and correspond to plane-polarized waves, while the integral curves of the third family are circles lying in the planes $u_3 = \text{const}$ with centres on the u_3 axis.

From the above-mentioned symmetry it also follows that $\lambda = \text{const}$ on any integral curve from the latter family, i.e. the corresponding wave retains its shape as it propagates. Using the terminology of magnetohydrodynamics, we shall call it a rotational wave. Simple waves were studied in [1–3] in cases of wave isotropy with functions F of certain special types.

When the wave anisotropy is small we shall express the elasticity potential $\Phi(u_i)$ as an isotropic part and a small correction

$$\Phi = F(r^2, u_3) + gp(u_i) \tag{1.3}$$

The first term on the right-hand side corresponds to the underlying isotropic part of the internal energy. The second term characterizes a small deviation of the medium from wave isotropy, $g \geq 0$ being a small number and $p(u_i)$ being, in general, an arbitrary function. The addition of anisotropy has little effect on the behaviour of the integral curves everywhere, except for small neighbourhoods of the points at which two of the eigenvalues of the matrix F_{ij} are identical. The origin in the u_i space, where the characteristic velocities of two transverse waves are equal to one another for $g = 0$, is always among such points. Depending on the form of F, other points and even whole surfaces with the above-mentioned property may appear.

Moreover, when there is anisotropy, even small, the integral curves of quasi-rotational waves are no longer circles, and, more importantly, the corresponding characteristic velocities c_θ along the curves are no longer constant. The extremum points of c_θ will be found on the integral curves of quasi-rotational waves. The deformation of the profile of such a wave is completely determined by the anisotropy ($g \neq 0$), even if it is small.

2. A study of non-linear waves of low intensity [4, 5] (near the origin in u_i space) revealed that anisotropy introduces more diversity and a qualitatively new behaviour of quasi-transverse waves. We would therefore expect the main effects caused by anisotropy to become apparent even for the model of an incompressible elastic medium, in which case there is no longitudinal deformation component ($u_3 = 0$), and, correspondingly, there are no quasi-longitudinal waves. The absence of the u_3 component makes it possible to continue the study in the phase plane u_1u_2.

For the medium under consideration, the elasticity potential (1.3) has the form

$$\Phi = F(r^2) + gp(u_1, u_2), \quad r^2 = u_1^2 + u_2^2$$

When there is no anisotropy ($g = 0$) the integral curves in the u_1u_2 plane form two orthogonal families: rays (plane-polarized waves) and circles (rotational waves). Their characteristic velocities (the eigenvalues of the matrix F_{ij}) are given by

$$\lambda_r^0 = \frac{d^2F}{dr^2} = f', \quad \lambda_\theta^0 = \frac{1}{r}\frac{dF}{dr} = \frac{f}{r}$$

Here and henceforth we use the notation $dF/dr = f(r)$, r and θ being polar coordinates in the u_1u_2 plane. The function $f(r)$ represents the dependence of the modulus of the shear stress

$$\sigma_\tau = \sqrt{\sigma_{31}^2 + \sigma_{32}^2} = \sqrt{(\partial F/\partial u_1)^2 + (\partial F/\partial u_2)^2} = f(r)$$

on the modulus of the shear strains $\varepsilon_\tau = (u_1^2 + u_2^2)^{1/2} = r$. Since $F(r^2)$ is an odd function of r, it follows that $f(r)$ is an even function. We shall assume that $f(r)$ has the expansion $f(r) = f'(0)r + \frac{1}{6}f'''(0)r^3 + \ldots$ in the neighbourhood of $r = 0$.

As has been mentioned above, small anisotropy may lead to a qualitative modification of the field of integral curves in the neighbourhood of any point at which the eigenvalues of the matrix F_{ij} are identical, i.e. any point where $d^0(r) \equiv \frac{1}{2}(f/r - f') = 0$. The point $r = 0$ is always a solution of the equation $d^0 = 0$. Besides, the equation may also have other solutions, which correspond to those points at which the ray passing through the origin is tangent to the graph of $f(r)$.

To fix our ideas, we will consider $f(r)$ to be of the form shown in Fig. 1, in which case $f'''(0) < 0$ and the function $d^0(r)$ has one non-zero root $r = r_*$. Moreover, we shall assume that $f'(r_* > 0)$. We note that if $r = r^\sim < r_*$, then the graph of $f(r)$ has a point of inflection, at which

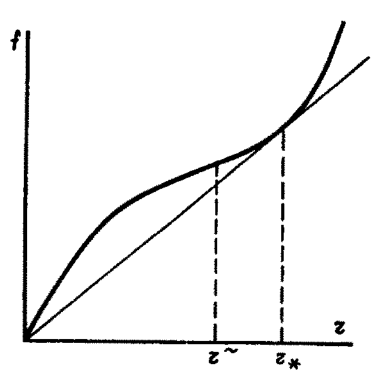

FIG. 1.

$f'(r^{\sim})=0$. We shall assume that $f(r)$ has no other points of inflection. The function $\lambda_r^0=f'(r)$ has a minimum at $r^{\sim}$. The sign of $f''(r)$ characterizes the convexity off the graph of $\sigma_\tau(\varepsilon_\tau)$. A function $f(r)$ similar to that in Fig. 1 can be found in materials which can undergo large elastic deformations and also in the case of active loading of many materials within the plasticity domain.

No further complications are required to consider waves in materials with opposite convexity or with two or more roots of $d^0(r)$.

Depending on the characteristic velocities λ_α ($\alpha=1, 2$), we shall distinguish between slow waves, for which $\lambda=\lambda_1$, and fast waves, for which $\lambda=\lambda_2$. Obviously, for a function $f(r)$ of the form chosen in Fig. 1, the radial waves are slow ($\lambda_r^0=\lambda_1$) and the rotational waves are fast ($\lambda_\theta^0=\lambda_2$) if $\lambda_2>\lambda_1$, and vice versa if $r>r_*$.

To study the integral curves and characteristic velocities in the phase plane u_1u_2 for $g\neq 0$, we will introduce an auxiliary system of coordinates y_α, which is a Cartesian system with origin at the given point r, θ and axes parallel and normal to the radius vector. In terms of these variables, the matrix $\Phi_{\alpha\beta}=\partial^2\Phi/\partial y_\alpha\partial y_\beta$ has the form $\Phi_{11}=f'+gp_{11}$, $\Phi_{12}=gp_{12}$, $\Phi_{22}=f/r+gp_{22}$, where $p_{\alpha\beta}=\partial^2 p/\partial y_\alpha\partial y_\beta$. The roots of the characteristic equation $|\Phi_{\alpha\beta}-\lambda\delta_{\alpha\beta}|=0$ give the eigenvalues

$$\lambda_{1,2}=f'+d+gp_{11}\mp\sqrt{d^2+g^2p_{12}^2},\quad d=d^0+\tfrac{1}{2}g(p_{22}-p_{11}) \tag{2.1}$$

The eigenvectors

$$\frac{dy_2}{dy_1}=\frac{d\mp\sqrt{d^2+g^2p_{12}^2}}{gp_{12}} \tag{2.2}$$

can be found from system (1.2).

The directions of the eigenvectors are undefined at the points where $\lambda_1=\lambda_2$. They are singular points, at which, according to (2.1), the two equations

$$d(r,\theta)=0,\quad p_{12}=0 \tag{2.3}$$

are satisfied simultaneously.

It is obvious that the singular points lie in domains in which the order of magnitude of $d^0(r)$ is equal to g. This is so, in particular, near the origin.

A study of the integral curves in the neighbourhood of the origin [4] revealed that there is a pair of singular points lying symmetrically about the origin, the distance between either of the points and the origin being of order $\sqrt{g}$. The other singular points lie in the vicinity of the critical circle $r=r_*$, where $d^0(r)=-\tfrac{1}{2}f_*''(r-r_*)+\ldots$. Their location is defined by the intersection of the lines $p_{12}=0$ and

$$r_{**}(\theta)=r_*+g(p_{22}-p_{11})/f_*'' \tag{2.4}$$

the latter being, obviously, close to the critical circle.

Far from the critical circle, i.e. for $|d^0|\gg g$, the direction of the eigenvector of the slow waves corresponding to λ_1 (the upper sign in (2.1) and (2.2)) is close to the radius vector for $r<r_*$, since $dy_2/dy_1\sim g$ in this region ($\lambda_1=\lambda_r$ being the quasi-radial wave). If $r>r_*$, then $dy_2/dy_1\sim g^{-1}$ for the same slow waves, and the eigenvector of this family is close to the direction of the y_2 axis, which is perpendicular to the radius vector ($\lambda_1=\lambda_\theta$ corresponds to a quasi-rotational wave). Thus the integral curves of slow waves rotate by an angle $\pi/2$, the process being concentrated in a narrow layer $\sim g$ near the critical circle.

At every point the integral curves of fast waves are perpendicular to the eigenvectors of the described family of slow waves. Thus, for $g\ll|d^0|$, they are quasi-rotational ($\lambda_2=\lambda_\theta$) in the domain $r<r_*$ and quasi-radial ($\lambda_2=\lambda_r$) for $r>r_*$. Moreover, the direction of the curves changes by an angle $\pi/2$ in a narrow zone near the critical circle.

The lines from either family rotate in the direction defined by the function dy_2/dy_1 given by (2.2). The numerator in (2.2) is always negative for the family of slow waves and positive for the family of fast waves. Thus, for $p_{12} > 0$, the integral curves of slow waves rotate to the right in the above-mentioned narrow zone as r increases, while for $p_{12} < 0$, the lines rotate to the left, along with the appropriate change of direction of the integral curves of fast waves. For $r < r_*$ the integral curves of slow waves and, for $r > r_*$, the integral curves of fast waves intersect the lines $p_{12} = 0$ along the radius vector $(dy_2/dy_1 = 0)$. For $r > r_*$ the integral lines of slow waves and, for $r < r_*$, the integral lines of fast waves intersect the lines $p_{12} = 0$ in a direction perpendicular to the radius vector.

An expansion of the functions (2.3) appearing in Eq. (2.2) up to the linear terms was used to study the behaviour of the integral curves near the singular points. In order to find the characteristic directions, a third-order equation was obtained, which can have either one or three real solutions, depending on the sign of $\partial p_{12}/\partial\theta$. If $\partial p_{12}/\partial\theta > 0$ at a singular point, then there is only one characteristic direction, which is close to the radius vector (at an angle $\alpha \approx g\partial p_{12}/\partial r$ to the vector), and which is tangent to a quasi-radial integral curve of slow waves for $r < r_*$ and to an integral curve of fast waves for $r > r_*$. But if $\partial p_{12}/\partial\theta < 0$ at the singular point, then, in addition to the above-mentioned radial directions, there are two more directions, the angle between either of which and the positive or negative direction of the y_2 axis (circle) being small.

The behaviour of the integral curves of slow and fast waves is shown in Fig. 2 in the case when $\partial p_{12}/\partial\theta > 0$ (a, b) and $\partial p_{12}/\partial\theta < 0$ (c, d).

As has already been mentioned, to study the deformation of the wave profile in time one must compute the derivatives of the characteristic velocities λ_α $(\alpha = 1, 2)$ along the corresponding integral curves. When differentiating $\lambda(u_\beta)$ we shall use a polar system of coordinates r, θ. As has been shown above, sufficiently far away from the origin and the critical circle, there is little difference between the integral curves and the rays and circles.

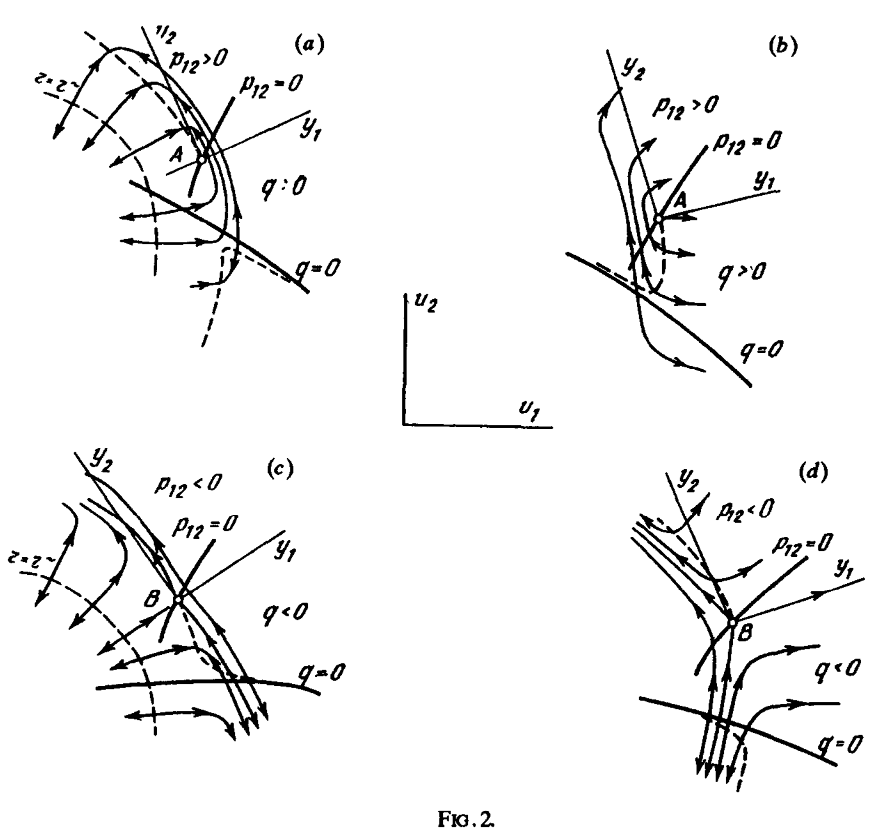

FIG. 2.

For the quasi-radial lines, we shall characterize the variation of λ by the derivative

$$\frac{d\lambda_r(r,\theta)}{dr}=\frac{\partial\lambda_r}{\partial r}+\frac{\partial\lambda_r}{\partial\theta}\frac{d\theta}{dr}$$

where $d\theta/dr=r^{-1}dy_2/dy_1$ is defined by the equation of the integral curve (2.2). We have

$$\frac{d\lambda_r}{dr}=\frac{\partial\lambda_r}{\partial r}=f''(r)+g\frac{\partial p_{11}}{\partial r} \tag{2.5}$$

with an error of order g^2.

Since, by assumption, f'' can vanish only at the point $r=r^{\sim}<r_*$, it follows that only for the slow quasi-radial waves an extremum of λ_1 exists on each integral curve, the extremum lying on a line close to the circle $r=r^{\sim}$ inside the critical circle.

For the quasi-rotational waves, we shall characterize the variation of λ by the derivative

$$\frac{d\lambda_\theta(r,\theta)}{d\theta}=\frac{\partial\lambda_\theta}{\partial r}\frac{dr}{d\theta}+\frac{\partial\lambda_\theta}{\partial\theta}$$

Taking into account that $|d^0|\gg g$ in the domain under consideration, one can neglect the terms containing higher powers of g when computing the derivatives. Thus one gets

$$\frac{d\lambda_\theta}{d\theta}=gq(r,\theta),\quad q(r,\theta)=\frac{\partial p_{22}}{\partial\theta}-\frac{p_{12}}{r} \tag{2.6}$$

It follows that, for $r-r_*\gg g$, the extremum points of λ_θ lie on the line $q(r,\theta)=0$ both for the fast waves (inside the critical circle $r=r_*$) and slow waves (outside the circle).

Using (2.5) and (2.6), the inequality $f''(r_*)>0$, and taking into account that every integral curve rotates according to the sign of p_{12} when intersecting a neighbourhood of the critical circle, one can draw the following conclusions. If q and p_{12} have opposite signs on a section of the critical circle, then, as a result of the intersection by the integral lines of slow waves (propagating almost along the radius for $r<r_*-O(g)$, and almost around the circle for $r>r_*-O(g)$), the sign of the derivative of λ_1 along the integral curve remains unchanged in the vicinity of the critical circle. In the same case, the sign of the derivative of λ_2 changes for the fast waves. If q and p_{12} have the same sign, then the sign of the derivative of λ changes for the slow waves and remains unchanged for the fast waves. If the sign of the derivative changes, it means that there is one extremum (or an odd number of extrema) of λ on the integral curve inside a neighbourhood of the critical circle. When the sign of the derivative is constant, no extremum points exist (or the number of extrema is even). A more detailed study of the behaviour of λ_α near the critical circle reveals that the possibilities mentioned in parentheses cannot be realized.

Indeed, the equation for the extremum line of λ_α can be represented in the form

$$\frac{d\lambda_\alpha}{dl}=\frac{1}{2}gp_{12}\left[\frac{qd}{p_{12}}+2f''+gA_1\pm\frac{(qd/p_{12}-2f''+gA)d+g^2B}{\sqrt{d^2+g^2p_{12}^2}}\right]=0$$

$$A=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{p_{22}-p_{11}}{2}\right)+\frac{\partial p_{12}}{\partial\theta},\quad A_1=A+\frac{\partial p_{11}}{\partial r} \tag{2.7}$$

$$B=\frac{\partial}{\partial\theta}\frac{p_{11}+p_{22}}{2}+\frac{\partial p_{12}}{\partial r}$$

Here dl is a parameter along an integral curve such that $dl\neq0$ in a neighbourhood of the critical circle for any non-zero element of the integral curve. On the extremum line, the number $d(r, \theta) = d^0 + \frac{1}{2}g(p_{22} - p_{11})$ introduced above can be determined from (2.7). It characterizes the distance between the extremum line of λ given by (2.7) and the line $d = 0$ measured along the radius, with $d > 0$ inside the line and $d < 0$ outside the line.

For $|d^0| \gg g$, the left-hand side of Eq. (2.7) takes the simplified forms (2.5) and (2.6). In the narrow zone under investigation, in which d is small, one can introduce the auxiliary extended variable $z = d/(gp_{12})$, for which $z \gg 1$ and $gz \ll 1$. Using an approximate expression for the root of the quadratic equation (2.7) and neglecting a number of terms in accordance with the latter inequalities, we obtain a solution of (2.7) in the domain where d is small: $d = gp_{12}(-f''/(gq))^{1/3}$. It is obvious that there is always a unique real root. For the fast waves (the plus sign in (2.7)), the solution acts in the domain where $d > 0$, i.e. $p_{12}q < 0$, and, for the slow waves (the minus sign), in the domain where $d < 0$, i.e. for $p_{12}q > 0$.

In Fig. 2 the extremum lines (2.7) are represented by the dashed lines, the arrows indicating the direction of increase of λ_α.

3. It is obvious that the number and location of singular points and extremum lines of λ_α depend very much on the form of $p(u_1, u_2)$. Expansions in terms of u_1 and u_2 were used to study the integral curves for $g \neq 0$ in a small neighbourhood of $r = 0$ [4, 5]. It turned out that if the expansion of $p(u_\alpha)$ contains quadratic terms, the function can be taken in the form $p = u_2^2 - u_1^2$ for small u_1 and u_2. We shall apply the same specific function $p(u_\alpha)$ to waves of finite intensity. This makes it possible to give a more illustrative representation of the behaviour of integral curves in the whole u_1, u_2 plane.

Now, suppose that the incompressible medium has the elasticity potential $\Phi = F(r^2) + \frac{1}{2}g(u_2^2 - u_1^2)$, where $dF/dr = f(r)$ is the function presented in Fig. 1. This means that one must set $p_{12} = \sin 2\theta$, $\frac{1}{2}(p_{22} - p_{11}) = \cos 2\theta$ and $p_{11} + p_{22} = 0$ in (2.1), (2.2) and (2.3). The integral curves turn out to be symmetric about the coordinate axes.

The singular points of the two families of integral curves coincide and line on the coordinate axes ($p_{12} = \sin 2\theta = 0$), their positions being given by $d^0(r) + g\cos 2\theta = 0$ (cf. (2.3)). This gives three pairs of points: the points C on the u_2 axis such that $u_1^C = 0$ and $u_2^C = \pm(-2g/f_0''')^{1/2}$, the points B on the u_2 axis such that $u_1^B = 0$ and $u_2^B = \pm(r_* - 2g/f_*'')$, and the points A on the u_1 axis such that $u_1^A = \pm(r_* + 2g/f_*'')$ and $u_2^A = 0$. The integral curves in the neighbourhood of C were studied in [4]. A study of the integral curves in the neighbourhood of B reveals there are three integral curves from each family passing through B, namely, the ray along the u_2 axis and two lines such that the angles $\pm\alpha = \pm(2g/r_* f_*'')^{1/2}$ between their directions and the u_1 axis are small. There is one integral curve passing through A, namely, the ray along the u_1 axis. The behaviour of the integral curves in the neighbourhood of A and B is the same as that shown in Fig. 2 to within a rotation.

The integral curves intersect the critical circle $r = r_*$ in directions parallel to the coordinate axes u_α. Near the circle $r = r_*$ the lines of either family rotate by an angle $\pi/2$ in the direction determined by the sign of $p_{12}/(\partial p_{12}/\partial\theta) = \mathrm{tg}\,2\theta$. The integral lines of slow and fast waves are depicted by the solid lines in Fig. 3.

The variation of the eigenvalues λ along the corresponding integral curves is defined by the derivative

$$\frac{d\lambda}{dl} = \frac{1}{2}g\sin 2\theta\left[f'' - \frac{6d^0}{r} \mp \frac{(f'' + 6d^0/r)(d^0 + g\cos 2\theta)}{[(d^0 + g\cos 2\theta)^2 + g^2\sin^2 2\theta]^{1/2}}\right] \tag{3.1}$$

Here $d^0 = d^0(r) = \frac{1}{2}(f/r - f')$, as before. By symmetry, it suffices to consider $d\lambda/dl$ only in the first quadrant, l being chosen so that it increases along the integral curve as r increases. It is clear that the sign of $d\lambda/dl$ changes on the coordinate axes, where $\sin 2\theta = 0$ (this corresponds to the equality $q = -3\sin 2\theta/r = 0$). Moreover, for the slow waves, the extrema of λ_1 form a line close to the circle $r = r^-$ ($f''(r^-) = 0$). The extremum line of λ_1 is represented by the dashed line in Fig. 3(a).

The fact that the lines $p_{12} = 0$ and $q = 0$ coincide introduces a certain symmetry and some

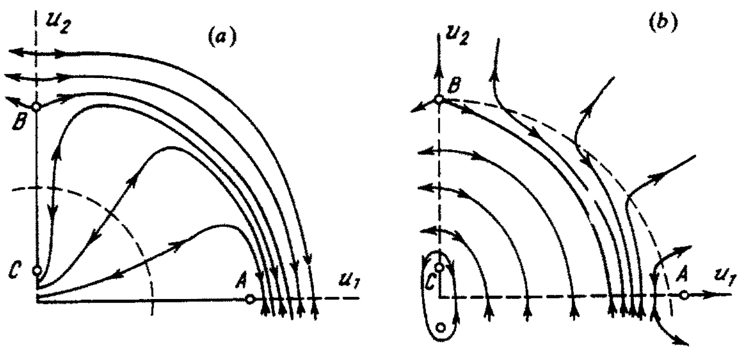

FIG. 3.

degeneracy into the problem. In particular, according to Sec. 2, since the signs of q and p_{12} are always different, an additional extremum of λ in the neighbourhood of the critical circle exists only for the fast waves. This portion of the extremum line of λ forms an oval symmetric about the coordinate axes, close to the critical circle, and contained completely within the circle. It passes through the singular point B and the point $u_1 = r_*$, $u_2 = 0$. The whole extremum line of λ_2 consists of the oval and those portions of the coordinate axes that are intersected by the integral curves of fast waves. It is represented by the dashed line in Fig. 3(b). The arrows indicate the direction of increase of λ_α on the integral lines.

4. We shall now consider the behaviour of the characteristic velocities and integral curves of Riemann waves in a compressible medium with small anisotropy, for which (1.3) holds. We will choose the axes of a local Cartesian system of coordinates y_i to be tangent to the coordinate lines of a cylindrical system with the z axis parallel to u_3. In this system $F_{12} = F_{32} = 0$, $F_{22} = F_1/r$ and $F_1 = \partial F/\partial r$. Here and henceforth the subscript 1 denotes differentiation with respect to $r = (u_1^2 + u_2^2)^{1/2}$, the subscript 3 denotes differentiation with respect to $z = u_3$, and the subscript 2 denotes differentiation along a tangent line to the circle of radius r.

In the region in which the eigenvalues of the matrix F_{ij} differ by a finite number, the integral curves for $g \neq 0$ are close to the corresponding integral curves obtained for $g = 0$. In Sec. 1 it was shown that, for $g = 0$, one of the families of integral curves in the u_i space consists of circles lying in the planes $u_3 = \text{const}$ (rotational waves), their characteristic velocity being given by the eigenvalue $\lambda_\theta^0 = F_1/r$. The other two families consist of plane-polarized waves with integral curves contained in the planes passing through the u_3 axis and having eigenvalues $\lambda_{r,z}^0 = \frac{1}{2}\{F_{11} + F_{33} \pm [(F_{11} - F_{33})^2 + 4F_{13}^2]^{1/2}\}$.

In order for the eigenvalues of two plane-polarized waves to be equal it is necessary for the two equalities $F_{11} = F_{33}$ and $F_{13} = 0$ to be satisfied simultaneously. In the general case, the equalities can be satisfied only at different points of the plane. To those points, provided they do not lie on the u_3 axis, there correspond circles in u_i space.

In the general case of a compressible medium, to investigate those wave properties that have been observed in the incompressible case, we shall assume that in some states it is possible for two eigenvalues to be equal in the domain under consideration in the compressible medium for $g = 0$, one of the eigenvalues corresponding to a rotational wave with $\lambda = \lambda_\theta^0$, and the other one to a plane-polarized wave which turns into a transverse wave with $\lambda = \lambda_r^0$ for small u_α. Since the eigenvalues of the matrix F_{ij} corresponding to the rotational and plane-polarized waves can be determined from independent equations, the requirement that the eigenvalues should be equal leads to a single equation, as opposed to the previous case. The equation defines a curve in the u_1, u_3 plane or a rotation-symmetric surface in u_i space. The equation has the form

$$D^0 \equiv (F_{11} - F_1/r)(F_{33} - F_1/r) - F_{13}^2 = 0$$

For the compressible medium under consideration, the rotation-symmetric surface with axis u_3 plays the same role as the critical circle $r = r_*$ in the case of an incompressible medium in Secs 2 and 3, $D^0(r, u_3)$ being similar to the function $d^0(r)$ used in those sections.

Consider the surface Σ^0 obtained by rotating an integral curve from the "first" family corresponding to λ_r^0 about the u_3 axis.

The surface can be thought of as being formed by the curves from the first family, each of the curves lying in a plane that passes through the u_3 axis. On the same surface lie the integral curves from the "second" family λ_θ^0, i.e. the circles $r = \text{const}$. The surface Σ^0 is tangent to the plane $u_3 = \text{const}$ at the point $u_1 = 0$, $u_2 = 0$, and, for small u_1 and u_2, it corresponds to quasi-transverse waves. The projections of the two families of integral curves onto the plane $u_3 = \text{const}$ coincide with the integral curves for the incompressible medium, i.e. lines and circles. Since the matrix F_{ij} is symmetric, the eigenvector of the third family λ_z^0 is normal to the eigenvectors of the other families, which implies that it is normal to Σ^0.

Thus, for $g = 0$, each of the surfaces Σ^0 consists of integral curves from the first and second families and plays the same role as the plane $u_3 = 0$ in the case of an incompressible medium. The only difference is that there are many surfaces Σ^0 and they fill the whole space u_i.

For a small $g \neq 0$, the integral curves from the first and second families remain close to the corresponding surfaces Σ^0 and the qualitative behaviour of their projections on to Σ^0 is the same as that of the integral curves in the plane $u_3 = 0$ in the case of an incompressible medium.

The first of these assertions follows from the fact that the integral curves are orthogonal to the eigenvector of the third family at every point, the eigenvector being almost normal to Σ^0 for any small $g \neq 0$. Everywhere away from the surface $D^0 = 0$ and the u_3 axis the integral curves of the two families under consideration are close to the integral curves for $g = 0$.

For small u_1 and u_2, the behaviour of the integral curves corresponding to the quasi-transverse waves is the same [4] as that of the integral curves for small u_1 and u_2 in the case of an incompressible medium (Secs 2 and 3). In the neighbourhood of the surface $D^0 = 0$, on which $\lambda_r^0 = \lambda_\theta^0$, the behaviour of the integral curves from either family undergoes qualitative modifications, while the curves remain close to their own surface Σ^0. The surface $D^0 = 0$ is rotationally symmetric, and its intersection with the surface Σ^0 is a circle, which is the analogue of the critical circle $r = r_*$ in the case of an incompressible medium.

In the neighbourhood of the surface $D^0 = 0$ the eigenvalues of the families under consideration can be represented by the formula

$$\lambda_{1,2} = \Phi_{22} + \tfrac{1}{2}\{D \mp [D^2 + 4g^2(p_{12}G - p_{23}\Phi_{13})^2]^{1/2}\}/G$$

$$D = (\Phi_{11} - \Phi_{22})G - \Phi_{13}^2, \quad G = \Phi_{33} - F_1/r$$

As in the case of an incompressible medium, the equality $\lambda_1 = \lambda_2$, which defines the position of the singular points, leads to the following two relations

$$D(r,\theta,u_3) = 0, \quad p_{12}G - p_{23}\Phi_{13} = 0 \tag{4.1}$$

One can see that the singular points form a line close to the surface $D^0 = 0$. On each surface Σ^0 there are isolated points, the intersections of the line (4.1) with the surface. The behaviour of the integral curves in the neighbourhood of a singular point is qualitatively the same as in the case of an incompressible medium.

If one intersects a narrow zone, where D^0 is of order g, without passing through a singular point, then λ_1 will remain distinct from λ_2 over the whole path. However, on one side of that narrow zone λ_1 corresponds to waves quasi-polarized in a plane, while, on the other side, it corresponds to quasi-rotational waves. This means that the integral curves corresponding to λ_1 rotate by 90° inside the zone, as in the case of an incompressible medium. The integral curves corresponding to λ_2 are orthogonal to the former ones and also rotate by 90°.

For the waves quasi-polarized in a plane the derivatives of λ along the corresponding integral curves are finite and close to their values for $g = 0$. For quasi-rotational waves the derivatives are small and are completely defined by the anisotropy of the medium

$$\frac{d\lambda_\theta}{d\theta}=g\left(\frac{\partial p_{22}}{\partial\theta}-\frac{p_{12}}{r}\right)\equiv gQ(r,\theta,u_3) \tag{4.2}$$

Equating (4.2) to zero, we obtain the equation of a surface, at the intersection with which λ_θ attains an extremum over its integral curve. The intersection of the surface with Σ^0 yields a curve similar to the curve $q=0$ in the case of an incompressible medium. Just as in the case of an incompressible medium, it can be shown that, in a small neighbourhood of the surface $D^0=0$, the sign of the derivative of λ also changes on an integral line from one of the families of integral lines corresponding to λ_1 and λ_2.

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